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Pointwise Unitary Automorphism Groups*

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Let (A, G, α) be a separable C^*-dynamical system in which G is abelian and A has continuous trace. The action α is said to pointwise unitary if for each π in $\hat{A}$ there is a covariant representation (π, U) of the system (A, G, α). It is shown that for pointwise unitary actions the spectrum $(A \times_\alpha G)^\wedge$ of the crossed product is a locally compact Hausdorff space and the restriction map Res, which takes $\pi \times U$ in $(A \times G)^\wedge$ to π, maps $(A \times G)^\wedge$ onto $\hat{A}$. Furthermore, Res induces a homeomorphism of $(A \times_\alpha G)^\wedge / \hat{G}$ onto $\hat{A}$. These results lead to a characterization of crossed products $A \times_\alpha G$ with continuous trace: when A has continuous trace and G acts freely on $\hat{A}$, this happens precisely when the action of G on $\hat{A}$ is also proper. This extends Green's characterization of free and proper transformation groups. Finally, it is shown that the results have applications to the problem of determining the dual topology on a locally compact group. © 1990 Academic Press, Inc.

Let (A, G, α) be a C^*-dynamical system in which G is abelian and A has continuous trace. As in [24], we say α is pointwise unitary if for each $\pi \in \hat{A}$ there is a covariant representation (π, U) of the system (A, G, α)—that is, if the induced action of G on $\hat{A}$ is trivial and all the Mackey obstructions vanish. If the implementing unitary representation U can locally be chosen of the form $\pi \circ u$ for the same strictly continuous map u of G into the multiplier algebra $M(A)$, then α is called locally unitary. In [24] it was shown that, for locally unitary α, the spectrum $(A \times_\alpha G)^\wedge$ of the crossed product is a locally trivial principal $\hat{G}$-bundle over $\hat{A}$, whose isomorphism class determines α up to exterior equivalence. Further, if A is stable every such

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bundle arises this way, so the locally unitary actions of G provide a C^*-algebraic realisation of the isomorphism theory of principal $\hat{G}$-bundles. The object of the present paper is to show that the pointwise unitary actions provide an analogous realisation for the free and proper actions of $\hat{G}$.

Our main result, then, is that virtually all of the main properties of locally unitary actions and their crossed products carry over to pointwise unitary actions provided one replaces locally trivial $\hat{G}$-bundles by free and proper $\hat{G}$-spaces. More precisely, we shall prove the following theorem.

THEOREM. *Let (A, G, α) be a separable C^*-dynamical system in which G is abelian, α is pointwise unitary, and A is a continuous-trace algebra with spectrum X.*

(1) *For each irreducible representation $\pi \times U$ of $A \times_\alpha G$,* $\operatorname{Res}(\pi \times U) = \pi$ *is irreducible; the restriction map* $\operatorname{Res}\colon (A \times_\alpha G)^\wedge \to X$ *is continuous, open, and surjective.*

(2) *The spectrum $(A \times_\alpha G)^\wedge$ is a locally compact Hausdorff space. The dual action of $\hat{G}$ on $(A \times_\alpha G)^\wedge$ is given by $\gamma \cdot (\pi \times U) = \pi \times \gamma U$; it is free and proper, and* Res *induces a homeomorphism of $(A \times_\alpha G)^\wedge / \hat{G}$ onto X. The $\hat{G}$-space* $\operatorname{Res}\colon (A \times_\alpha G)^\wedge \to X$ *determines α up to exterior equivalence.*

(3) *There is an isomorphism of $A \times_\alpha G$ onto the pull-back* $\operatorname{Res}^* A = C_0((A \times_\alpha G)^\wedge) \otimes_{C(X)} A$ *of* [26], *which carries the dual action $\hat{\alpha}$ into* $\operatorname{Res}^*(\mathrm{id})$.

(4) *Suppose that A is stable and $g\colon E \to X$ is the orbit map for a free and proper action of $\hat{G}$. Then there is a pointwise unitary action α of G on A such that there is a $\hat{G}$-equivariant homeomorphism h of $(A \times_\alpha G)^\wedge$ onto E with* $\operatorname{Res} = g \circ h$.

For locally unitary actions (1), (2), (3), and (4) were proved in [24, Proposition 2.1; 24, Theorem 2.2 and Proposition 2.5; 26, Proposition 1.5; and 24, Theorem 3.8], respectively. Example 1.2 of [24] shows that there are pointwise unitary actions which are not locally unitary, so our present theorem is a non-vacuous extension of these results.

The idea of our proof of this theorem is quite simple. By a theorem of Rosenberg, when G is compactly generated every pointwise unitary action is locally unitary, and the result is therefore known. On the other hand, when G is discrete it is possible to prove the theorem by exploiting the compactness of the dual group. Since every locally compact abelian group has a compactly generated subgroup H with G/H discrete, we tackle the general problem by trying to patch these two special cases together. Unfortunately, our method of doing this requires that we also consider the H-inner actions of Green [12], which we now discuss.

Suppose $\alpha: G \to \operatorname{Aut} A$ is given on a normal subgroup by a twisting map $\mathscr{I}: H \to UM(A)$— that is, $\mathscr{I}$ is a strictly continuous homomorphism such that $\operatorname{Ad} \mathscr{I} = \alpha|_H$ and $\mathscr{I}(shs^{-1}) = \alpha_s(\mathscr{I}(h))$ for $s \in G$, $h \in H$; following [19], we shall say $(A, G, \alpha, \mathscr{I})$ is an *H-inner system.* The *restricted crossed product* $A \times_{\alpha|_H} G$ by an H-inner action is the quotient of $A \times_\alpha G$ by the common kernel of the representations $\pi \times U$ such that (π, U) preserves $\mathscr{I}$, in the sense that $U|_H = \pi \circ \mathscr{I}$ [3, 12]. (These are perhaps more commonly known as twisted crossed products, but we avoid this to minimise confusion with the more general twisted crossed products of [20], which we shall use later.) The key to the proof of our theorem is a result of Green [12, Proposition 1], which asserts that if $\alpha: G \to \operatorname{Aut} A$ and H is normal in G, there is an H-inner action β of G on $A \times_\alpha H$ such that $A \times_\alpha G$ is isomorphic to $(A \times_\alpha H) \times_{\beta|H} G$. Since restricted crossed products by H-inner actions behave very much like ordinary crossed products by G/H, we shall try to prove our main result by extending the argument for discrete G to cover H-inner action, where G/H is discrete and then using Green's decomposition to get to the general case. Thus we begin by developing the basic facts for pointwise unitary H-inner actions; we could presumably do the whole thing in this generality, but as we shall see in Section 2 (b) there is a simple trick for extending results to this case.

The proof of our main theorem is the content of our first section. In Section 2, we consider some generalizations and extensions of our results. As we have how carried over essentially all of the known theory of locally unitary actions to pointwise unitary actions, one can now hope to replace locally unitary actions everywhere they are used. We begin Section 2 by showing now the results of [25, Section 2] on actions with constant isotropy can easily be so extended. We then show how the stabilisation trick of [20, Section 3] enables us to immediately extend most of our main theorem to the twisted actions of [20]; we shall need this extension for some of our applications. We finish Section 2 with an example which shows that the continuous trace hypothesis on A is necessary; this is certainly one difference between our present results and those of [24], where it was enough to assume $\hat{A}$ Hausdorff.

In our final section we present two applications. The first is a characterisation of crossed products $A \times_\alpha G$ with continuous trace: when A has continuous trace and G acts freely on $\hat{A}$, this happens precisely when the action of G on $\hat{A}$ is also proper. This extends, at least in the case of abelian G, Green's elegant characterisation of free and proper transformation groups [11], but as far as we know, this is the first such result for non-commutative A (see [8]). Our second application concerns the old problem of determining the dual topology on a locally compact group G when the representations themselves are described by the Mackey machine. A program to do this was developed by, among others, Fell and Baggett

[1]: their results and conjectures were concerned mainly with the local problem of describing the limit points of nets in $\hat{G}$ obtained by inducing from varying stabilisers. Later a detailed and systematic collection of conjectures was made by Schochetman [30], in which he also aimed to describe the global structure of the parts of $\hat{G}$ where induction was not a necessary part of the machine. We shall review these conjectures in the light of our own and other recent work on the topology of the spectrum of crossed products; roughly speaking, our thesis is that they should be amended to allow for non-trivial bundles and spaces. We do obtain some positive results here, based on our work on twisted actions in Section 2, but these are only fragmentary, and our main intention is to point out that the phenomena we have been studying are relevant to this problem.

1. Crossed Products by Pointwise Unitary Actions

Our object in this section is to prove the theorem stated in the introduction concerning the topology of $(A \times_\alpha G)^\wedge$ and the structure of $A \times_\alpha G$ when α is pointwise unitary. We begin slightly more generally by considering H-inner actions of G where G/H is abelian, and for this we shall need appropriate notions of pointwise and local unitarity.

Suppose $(A, G, \alpha, \mathscr{I})$ is an H-inner system with A a continuous-trace algebra. We say α is *pointwise unitary relative to* $\mathscr{I}$ if for each $\pi \in \hat{A}$ there is a covariant representation (π, U) of (A, G, α) on $\mathscr{H}_\pi$ which preserves the twist $\mathscr{I}$ (i.e., has $U|_H = \pi \circ \mathscr{I}$). Similarly, α is *locally unitary relative to* $\mathscr{I}$ if for each $\pi \in \hat{A}$ there are a neighbourhood $\mathscr{U}$ of π and a strictly continuous map $u: G \to UM(A)$ such that for $\rho \in \mathscr{U}$, $(\rho, \rho \circ u)$ is a covariant representation of (A, G, α) which preserves $\mathscr{I}$. Obviously, such actions are pointwise or locally unitary in the usual sense, and conversely we have:

Lemma 1.1. *Suppose that α is pointwise unitary and that G is either abelian or a semi-direct product of H by G/H. Then α is pointwise unitary relative to $\mathscr{I}$.*

Proof. Let $\pi \in \hat{A}$. There is a representation $V: G \to \mathscr{U}(\mathscr{H}_\pi)$ such that

$$V_s \pi(a) V_s^* = \pi(\alpha_s(a)) \qquad \text{for all} \quad a \in A, \quad s \in G.$$

For $h \in H$ we have $\operatorname{Ad} V_h = \operatorname{Ad} \pi(\mathscr{I}_h)$ on $\pi(A) = \mathscr{K}(\mathscr{H}_\pi)$, so there is a scalar $\gamma(h)$ such that $\gamma(h) V_h = \pi(\mathscr{I}_h)$. Both V and $\pi \circ \mathscr{I}$ are continuous homomorphisms, and hence $\gamma \in \hat{H}$. Under either hypothesis on G, γ extends to a character χ of G, and then $U = \chi V$ is a representation of G in $\mathscr{H}_\pi$ with $U|_H = \pi \circ \mathscr{I}$ and which implements α in the representation π.

We now turn to the problem of describing $(A \times_{\alpha|H} G)^\wedge$ when α is pointwise unitary relative to $\mathcal{I}$. Recall that $A \times_{\alpha|H} G$ can be concretely constructed as the C^*-enveloping algebra of the algebra $C_c(G, A, \mathcal{I})$ of continuous functions $f: G \to A$ such that $f(hs) = f(s)\mathcal{I}_h^*$ for $h \in H$, $s \in G$, and such that $sH \to \|f(s)\|$ has compact support in G/H (see [3, 12]). If (π, U) is a covariant representation of (A, G) which preserves $\mathcal{I}$ then the corresponding representation $\pi \times U$ of $A \times_{\alpha|H} G$ satisfies

$$\pi \times U(f) = \int_{G/H} \pi(f(s)) U_s d(sH) \qquad \text{for} \quad f \in C_c(G, A, \mathcal{I}).$$

When G/H is abelian, there is a dual action $\hat{\alpha}$ of $(G/H)^\wedge$ on $A \times_{\alpha|H} G$ such that

$$\hat{\alpha}_\gamma(f)(s) = \overline{\gamma(s)} f(s) \qquad \text{for} \quad f \in C_c(G, A, \mathcal{I}).$$

PROPOSITION 1.2. *Suppose $(A, G, \alpha, \mathcal{I})$ is an H-inner C^*-dynamical system with G/H abelian, A continuous-trace, and α pointwise unitary relative to $\mathcal{I}$. If $\pi \times U \in (A \times_{\alpha|H} G)^\wedge$, then π is irreducible and the map $\mathrm{Res}: \pi \times U \to \pi$ is a continuous open surjection of $(A \times_{\alpha|H} G)^\wedge$ onto $\hat{A}$. The dual action of $(G/H)^\wedge$ on $(A \times_{\alpha|H} G)^\wedge$ is given by $\gamma \cdot (\pi \times U) = \pi \times \gamma U$, and for $\pi \times U \in (A \times_{\alpha|H} G)^\wedge$*, we have

$$\mathrm{Res}^{-1}\{\pi\} = \{\pi \times \gamma U : \gamma \in (G/H)^\wedge\}.$$

Proof. We begin as in the proof of [24, Proposition 2.1]. If $\pi \times U \in (A \times_{\alpha|H} G)^\wedge$, then because A is type I we can choose unitary operators $V_t \in \pi(A)''$ which implement α_t in the representation π. It follows from the covariance of (π, U) that $V_t^* U_t \in \pi(A)'$. Suppose that P belongs to the centre $\pi(A)'' \cap \pi(A)'$ of $\pi(A)''$. Then for each $t \in G$,

$$PU_t = PV_t V_t^* U_t = V_t P(V_t^* U_t) = V_t V_t^* U_t P = U_t P,$$

so that P commutes with the range of U as well as the range of π, and hence belongs to $((\pi \times U)(A \times_{\alpha|H} G))' = \mathbf{C}1$. We deduce that π is a factor representation, type I since A is, and can therefore be realised as $\rho \otimes 1$ acting on $\mathcal{H}_\rho \otimes \mathcal{H}$ for some $\rho \in \hat{A}$. Now the assumption on α ensures the existence of a unitary representation $W: G \to \mathcal{U}(\mathcal{H}_\rho)$ which implements α in the representation ρ and furthermore satisfies $W_h = \rho(\mathcal{I}_h)$ for every $h \in H$. So $(W_t \otimes 1)^* U_t$ belongs to $\pi(A)' = 1 \otimes B(\mathcal{H})$ and hence has the form $1 \otimes Y_t$. The condition $W_t \otimes 1 \in (1 \otimes B(\mathcal{H}))'$ implies that Y is a representation of G, and as $(\pi, U) = (\rho \otimes 1, W \otimes Y)$ is irreducible, Y must be too. Using now that $\rho(\mathcal{I}_h) = W_h$ and $\pi(\mathcal{I}_h) = U_h$, we further see that for $h \in H$

$$Y_h = (W_h \otimes 1)^* U_h = (\rho(\mathcal{I}_h) \otimes 1)^* \pi(\mathcal{I}_h) = 1,$$

so Y is really a representation of G/H. But G/H is abelian, so $\mathscr{H}$ must be one-dimensional and $\pi=\rho$ is irreducible.

We therefore have a well-defined restriction map Res: $(A \times_{\alpha|H} G)^\wedge \to \hat{A}$, as claimed. It is surjective because α is pointwise unitary relative to $\mathscr{I}$ and continuous because restriction always is [12, Proposition 9]. That Res is open follows from a theorem of Williams [32, Theorem 2.1]; it is easy to see that the map β he constructs agrees with our restriction map. To see that the dual action has the required property is also easy:

$$\pi \times U(\hat{\alpha}_\gamma^{-1}(f)) = \int_{G/H} \pi(\gamma(s) f(s))\, U_s\, d(sH) = (\pi \times \gamma U)(f)$$

for $f \in C_c(G, A, \mathscr{I})$. Finally, suppose both (π, U) and (π, V) are irreducible representations of (A, G) preserving $\mathscr{I}$. The irreducibility of π implies that each U_s is a multiple $\gamma(s) V_s$ of V_s, and an easy calculation shows $\gamma \in \hat{G}$; in fact $\gamma \in H^\perp$ because both U and V preserve $\mathscr{I}$ and hence agree on H. Thus $\mathrm{Res}^{-1}\{\pi\}$ has the required form, and this completes the proof of Proposition 1.2.

Our next goal is to prove that $(A \times_{\alpha|H} G)^\wedge$ is Hausdorff when G/H is discrete. For this we shall not need the next two results in their full generality, but they may be of some independent interest; they generalise, respectively, [25, Theorem 0.11; 29, Corollary 2.2] to H-inner actions.

PROPOSITION 1.3. *Let $(A, G, \alpha, \mathscr{I})$ be a separable H-inner system in which A is a continuous-trace algebra with spectrum X, and suppose λ is an action of G/H on A such that $\alpha \circ \lambda^{-1}$ consists of inner automorphisms. Give the unitary group $U\mathscr{M}(A)$ the Borel structure generated by the strict topology. Then there is a Borel map $u: G \to U\mathscr{M}(A)$ such that*

(i) $\mathrm{Ad}\, u_s = \alpha_s \circ \lambda_s^{-1}$ *for* $s \in G$;

(ii) $u_{hs} = \mathscr{I}_h u_s$ *for* $h \in H$, $s \in G$.

Give $C(X, \mathbf{T})$ the Borel structure it inherits as a subgroup of $U\mathscr{M}(A)$ and the G-structure induced by the action α on $X = \hat{A}$; note that since $\alpha|_H = \mathrm{Ad}\,\mathscr{I}$ consists of inner automorphisms, H acts trivially on X. The cocycle $\omega \in Z^2(G, C(X, \mathbf{T}))$ defined by

(iii) $\omega(s, t) u_{st} = u_s \lambda_s(u_t)$

is constant on H-cosets, and its class in $H^2(G/H, C(X, \mathbf{T}))$ vanishes if and only if there is a continuous map $w: G \to U\mathscr{M}(A)$ such that $w|_H = \mathscr{I}$, $\mathrm{Ad}\, w = \alpha \circ \lambda^{-1}$, and $w_{st} = w_s \lambda_s(w_t)$.

Proof. The map $\alpha \circ \lambda^{-1}: G \to \mathrm{Inn}\, A$ is continuous for the quotient Polish topology on $\mathrm{Inn}\, A = U\mathscr{M}(A)/C(X, \mathbf{T})$ by [25, Corollary 0.2], and

$U\mathcal{M}(A) \to U\mathcal{M}(A)/C(X, \mathbf{T})$ has a Borel section, so we can find a Borel map $v: G \to U\mathcal{M}(A)$ such that $\mathrm{Ad}\, v = \alpha \circ \lambda^{-1}$. We fix a Borel section $c: G/H \to G$ and define

$$u_s = \mathscr{I}_{sc(Hs)^{-1}} v_{c(Hs)}.$$

Note that since $H \subseteq \ker \lambda$, $\lambda_s = \lambda_{c(Hs)}$ and

$$\begin{aligned}
\mathrm{Ad}\, u_s &= \mathrm{Ad}\, \mathscr{I}_{sc(Hs)^{-1}} \circ \mathrm{Ad}\, v_{c(Hs)} \\
&= \alpha_{sc(Hs)^{-1}} \circ [\alpha_{c(Hs)} \circ \lambda_{c(Hs)}^{-1}] \\
&= \alpha_s \circ \lambda_{c(Hs)}^{-1} \\
&= \alpha_s \circ \lambda_s^{-1},
\end{aligned}$$

so (i) holds. Condition (ii) is easy to check since $c(Hhs) = c(Hs)$. The map ω defined by (iii) is certainly Borel and takes values in $C(X, \mathbf{T}) = ZU\mathcal{M}(A)$; it is a cocycle by the usual argument using the associative law in G. For $h, k \in H$ we have

$$\begin{aligned}
\omega(hs, kt)1 &= u_{hskt}^* u_{hs} \lambda_{hs}(u_{kt}) \\
&= u_{h(sks^{-1})st}^* \mathscr{I}_h u_s \lambda_s(\mathscr{I}_k u_t) \\
&= (\mathscr{I}_h \mathscr{I}_{sks^{-1}} u_{st})^* \mathscr{I}_h u_s \lambda_s(\mathscr{I}_k) \lambda_s(u_t) \\
&= u_{st}^* \mathscr{I}_{sks^{-1}}^* [u_s \lambda_s(\mathscr{I}_k) u_s^*] u_s \lambda_s(u_t) \\
&= u_{st}^* \mathscr{I}_{sks^{-1}}^* [\alpha_s(\mathscr{I}_k)] u_s \lambda_s(u_t) \\
&= u_{st}^* u_s \lambda_s(u_t) \\
&= \omega(s, t)1,
\end{aligned}$$

so $\omega \in Z^2(G/H, C(X, \mathbf{T}))$. If we have a crossed homomorphism (1-cocycle) $w: G \to U\mathcal{M}(A)$ with $\mathrm{Ad}\, w = \alpha \circ \lambda^{-1}$, then in particular $\mathrm{Ad}\, w = \mathrm{Ad}\, u$, so there is a Borel map $d: G \to C(X, \mathbf{T})$ such that $d_s w_s = u_s$, and we get $\omega(s, t) = d_s d_t d_{st}^*$. Further, if $h \in H$ then

$$d_{hs} = w_{hs}^* u_{hs} = (\mathscr{I}_h w_s)^* (\mathscr{I}_h u_s) = w_s^* u_s = d_s,$$

so d is constant on H cosets and $w = 0$ in $H^2(G/H, C(X, \mathbf{T}))$. Conversely, if $\omega(s, t) = d_s d_t d_{st}^*$ for some Borel $d: G/H \to C(X, \mathbf{T})$, then $w_s = d_s^* u_s$ is a Borel map with the right properties which is automatically continuous since $U\mathcal{M}(A)$ is Polish.

PROPOSITION 1.4. *Let $(A, G, \alpha, \mathscr{I})$ be a separable H-inner system with A continuous-trace and α pointwise unitary relative to $\mathscr{I}$. Suppose that*

$H^2(G/H, \mathbf{T})$ is Hausdorff and that $(G/H)_{ab} = (G/H)/[G/H, G/H]$ is compactly generated. Then α is locally unitary relative to $\mathscr{I}$.

Remark. Both hypotheses on G/H are satisfied if G/H is compactly generated and abelian [18, Theorem 7]. For other conditions which imply these hypotheses, see the statement of [29, Theorem 2.1].

Proof. We follow the proof of [29, Corollary 2.2]. This is a local problem, so we may as well suppose that $X = \hat{A}$ is compact. We write ε_y for the representation of A corresponding to $y \in X$. The automorphisms α_s all fix the spectrum and $\alpha|_H$ consists of inner automorphisms, and composing with the quotient map therefore gives a continuous homomorphism of G/H into the discrete abelian group $\mathrm{Aut}_{C(X)} A/\mathrm{Inn}\, A \hookrightarrow H^2(X, Z)$. It follows that by shrinking X we can suppose that α consists of inner automorphisms (this uses the compact generation of G/H). If we now take u and ω as in the previous proposition, with λ trivial, and evaluate them at a point y in X, we see that $\omega(\cdot, \cdot)(y) \in Z^2(G/H, \mathbf{T})$ is the Mackey obstruction to implementing α in the representation ε_y by a unitary representation $\varepsilon_y \circ \tau$. The pointwise unitary hypothesis therefore says precisely that the class of each $\omega(\cdot, \cdot)(y)$ in $H^2(G/H, \mathbf{T})$ is 0—in other words, $[\omega] \in H^2(G/H, C(X, \mathbf{T}))$ is pointwise trivial in the sense of [29, Theorem 2.1]. By that theorem, therefore, each point x in X has a neighbourhood M such that the restriction of ω to $C(M, \mathbf{T})$ is trivial in $H^2(G/H, C(M, \mathbf{T}))$, By Proposition 1.3, $[\omega|_{C(M, \mathbf{T})}]$ is the only obstruction to implementing $\alpha|_{A|_M}$ by a unitary group extending $\mathscr{I}$, and α is therefore locally unitary relative to $\mathscr{I}$.

PROPOSITION 1.5. *Let $(A, G, \alpha, \mathscr{I})$ be a separable H-inner C^*-dynamical system with G abelian, G/H discrete, A continuous-trace and α pointwise unitary relative to $\mathscr{I}$. Then $(A \times_{\alpha|H} G)^\wedge$ is Hausdorff.*

Proof. Since $\mathrm{Res} = (A \times_{\alpha|H} G)^\wedge \to \hat{A}$ is continuous and $\hat{A}$ is Hausdorff it will suffice to show that if $\pi_1, \pi_2 \in (A \times_{\alpha|H} G)^\wedge$ satisfy $\mathrm{Res}\, \pi_1 = \mathrm{Res}\, \pi_2$, $\pi_1 \neq \pi_2$, then π_1 and π_2 have disjoint neighbourhoods in $(A \times_{\alpha|H} G)^\wedge$. Since this property is local in $\hat{A}$ and unchanged by stabilizing, we may as well suppose that $A = C(X, \mathscr{K})$ with X compact. Let $\varepsilon_x = \mathrm{Res}\, \pi_1 = \mathrm{Res}\, \pi_2$ and suppose $u: G \to \mathscr{U}(\mathscr{H}_x)$ implements α in the representation ε_x and satisfies $u|_H = \mathscr{I}(\cdot)(x)$. Then by Proposition 1.2 there exist $\gamma_i \in H^\perp$ such that $\pi_i \sim \varepsilon_x \times \gamma_i u$. We shall from now on identify $\pi_i = \varepsilon_x \times \gamma_i u$. As we are assuming $\pi_1 \neq \pi_2$, we get $\gamma_1 \neq \gamma_2$ and so we can find disjoint open sets F_1 and F_2 in $(G/H)^\wedge \cong H^\perp$ such that $\gamma_i \in F_i$. We may as well take each F_i to be a basic set in G/H, i.e., there are K_i compact ($=$finite in G/H) and $\varepsilon_i > 0$, such that

$$F_i = \{\chi \in (G/H)^\wedge \mid |\chi(s) - \gamma_i(s)| < \varepsilon_i \ \forall s \in K\}.$$

We may further suppose $\varepsilon_1 = \varepsilon_2 = \varepsilon < \sqrt{2}$, and

$$K_1 = K_2 = \{g_1 H, ..., g_J H\} = K,$$

where $g_1, g_2, ..., g_J$ lie in distinct H cosets in G. Let N_1 be the subgroup of G/H generated by K, and let N be its inverse image in G. By Proposition 1.7 there is a neighbourhood M of x and a strictly continuous homomorphism $n: N \to C(M, \mathscr{U}(\mathscr{H}))$ such that

$$n_h(y) = \mathscr{I}(h)(y) \qquad \forall h \in H, \quad \forall y \in M$$
$$\alpha_s(a)\, y = n_s(y) a(y) n_s(y)^* \qquad \forall y \in M, \quad \forall s \in H.$$

Without loss of generality we may assume M to be compact and replace A by $C(M, \mathscr{K})$. We now choose a unit vector ξ in $\mathscr{H}$, and define a_j in A by

$$a_j(y) = \xi \otimes \overline{n_{g_j}(y)\xi}.$$

Now define $f_j: G \to A$ by

$$f_j(g) = \begin{cases} a_j \mathscr{I}(h)^{-1} & \text{if } g \in g_j H, \quad g = hg_j \\ 0 & \text{if } g \notin g_j H. \end{cases}$$

Then for $k \in H$ we have

$$\begin{aligned} f_j(kg) &= \begin{cases} a_j \mathscr{I}(h_1)^{-1} & \text{if } kg \in g_j H, \quad kg = h_1 g_j \\ 0 & \text{if } kg \notin g_j H \end{cases} \\ &= \begin{cases} a_j \mathscr{I}(kh)^{-1} & \text{if } g \in g_j H, \quad g = hg_j \\ 0 & \text{if } g \notin g_j H \end{cases} \\ &= \begin{cases} a_j \mathscr{I}(h)^{-1} \mathscr{I}(k)^{-1} & \text{if } g \in g_j H, \quad g = hg_j \\ 0 & \text{if } g \notin g_j H \end{cases} \\ &= f_j(g) \mathscr{I}(k)^{-1} \end{aligned}$$

so each $f_j \in C_c(G, A, \mathscr{I})$ (as in [12, p. 197]). We extend $g_1, ..., g_J$ to a (countable) set (g_j) of coset representatives for G/H. For any covariant representation $\pi \times v$ of (A, G, α) which preserves the twist $\mathscr{I}$, and any f in $C_c(G, A, \mathscr{I})$, the function $s \mapsto \pi(f(s))v_s$ from G to A is constant on H cosets, and the corresponding representation of $A \times_{\alpha|H} G$ is given by

$$(\pi \times v)(f) = \int_{G/H} \pi(f(s)) v_s \, ds = \sum_j \pi(f(g_j)) v_{g_j}.$$

In particular, for $1 \leqslant j \leqslant J$ we have

$$(\varepsilon_y \times v)(f_j) = \sum_k f_j(g_k)(y) v_{g_k} = f_j(g_j)(y) v_{g_j}$$
$$= a_j(y) v_{g_j} = (\xi \otimes \overline{n_{g_j}(y)\xi})\, v_{g_j}. \qquad (*)$$

We define

$$M_i = \left\{ \varepsilon_y \times v \in (A \times_{\alpha|H} G)^\wedge \mid y \in M \text{ and } \exists v \in \mathscr{H},\ \|v\| = 1, \right.$$
$$\left. \text{such that } |((\varepsilon_y \times v)(f_j) v \mid v) - (\pi_i(f_j)\xi \mid \xi))| < \frac{\varepsilon}{\sqrt{2}} \text{ for } 1 \leqslant j \leqslant J \right\}.$$

Then M_i is a basic open neighbourhood of π_i in $\operatorname{Res}^{-1}(M)$. Now suppose $\varepsilon_y \times v \in M_1 \cap M_2$, and v_1, v_2 are the corresponding unit vectors such that $((\varepsilon_y \times v)(f_j) v_i \mid v_i)$ is close to $(\pi_i(f_j)\xi \mid \xi)$. Then from $(*)$ above,

$$((\varepsilon_y \times v)(fj) v_i \mid v_i) = ((\xi \otimes \overline{n_{g_j}(y)\xi}) v_{g_j} v_i \mid v_i)$$
$$= (v_{g_j} v_i \mid n_{g_j}(y)\xi)(\xi \mid v_i).$$

Since $v|_N$ and $n(y)$ both implement $\alpha|_N$ in the representation ε_y of $C(M, \mathscr{K})$, there is a γ in $\hat{N}$ such that $v|_N = \gamma n(y)$. Further

$$v|_H = \varepsilon_y \circ \mathscr{I} = n(y)|_H$$

so $\gamma \in H^\perp$. Then

$$((\varepsilon_y \times v)(f_j)\, v_i \mid v_i) = (\gamma(g_j) n_{g_j}(y) v_i \mid n_{g_j}(y)\xi)(\xi \mid v_i)$$
$$= \gamma(g_j)(v_i \mid \xi)(\xi \mid v_i)$$
$$= \gamma(g_j) |(\xi \mid v_i)|^2$$

and

$$(\pi_i(f_j)\xi \mid \xi) = ((\varepsilon_x \times \gamma_i u)(f_j)\xi \mid \xi)$$
$$= (\xi \otimes \overline{n_{g_j}(x)\xi}\, \gamma_i(g_j) u_{g_j} \xi \mid \xi)$$
$$= \gamma_i(g_j)(u_{g_j}\xi \mid n_{g_j}(x)\xi)(\xi \mid \xi).$$

As before, $n(x)$ and $u|_N$ both implement $\alpha|_N$ in the representation ε_x, and both $(\varepsilon_x, n(x))$ and $(\varepsilon_x, u|_N)$ respect $\mathscr{I}$, so there is a λ in $H^\perp \cap \hat{N}$ such that $\lambda n(x) = u|_N$, and then

$$(\pi_i(f_j)\xi \mid \xi) = \gamma_i(g_j)\lambda(g_j)(n_{g_j}(x)\xi \mid n_{g_j}(x)\xi)(\xi \mid \xi)$$
$$= \gamma_i(g_j)\lambda(g_j).$$

We therefore have

$$|\gamma(g_j)|\,|(\xi\,|\,v_i)|^2-\gamma_i(g_j)\lambda(g_j)|<\frac{\varepsilon}{\sqrt{2}} \qquad \text{for} \quad 1\leqslant j\leqslant J, \quad i=1,2. \quad (**)$$

LEMMA 1.6. *Suppose* $r\in[0,1]$, $|z|=|w|=1$, *and* $|zr-w|<\delta<1$. *Then* $|z-w|<\sqrt{2}\,\delta$.

Proof. That $|zr-w|<\delta$ says that the distance from w to the ray $[0,1]z$ is less than δ. Call the actual distance c, then $c<\delta$. Now

$$\begin{aligned}|z-w|&=\sqrt{c^2+(1-\sqrt{1-c^2})^2}\\&=\sqrt{c^2+1+1-c^2-2\sqrt{1-c^2}}\\&=\sqrt{2}\sqrt{1-\sqrt{1-c^2}}.\end{aligned}$$

But $c^2-(1-\sqrt{1-c^2})=\sqrt{1-c^2}-(1-c^2)\geqslant 0$ for $c<1$, hence $\sqrt{1-\sqrt{1-c^2}}\leqslant c$, so $|z-w|\leqslant\sqrt{2}\,c<\sqrt{2}\,\delta$, as required.

Returning to the proof of Proposition 1.5, we see that from the lemma and $(**)$ we deduce that

$$|\gamma(g_j)-\gamma_i(g_j)\lambda(g_j)|<\varepsilon \qquad \text{for} \quad 1\leqslant j\leqslant J \quad \text{and} \quad i=1,2,$$

which says that

$$|(\gamma\bar{\lambda})(g_j)-\gamma_i(g_j)|<\varepsilon \qquad \text{for} \quad 1\leqslant j\leqslant J \quad \text{and} \quad i=1,2.$$

The character $\gamma\bar{\lambda}$ extends to a character χ of G which belongs to $H^{\perp}$ since both γ and λ do. Then χ belongs to $F_1\cap F_2=\varnothing$, a contradiction. Hence no such $\varepsilon_y\times v$ as assumed exists in $M_1\cap M_2$, and we conclude that M_1 and M_2 are disjoint.

THEOREM 1.7. *Suppose* (A, G, α) *is a separable dynamical system with* A *continuous-trace,* G *abelian, and* α *pointwise unitary. Then* $(A\times_\alpha G)^\wedge$ *is Hausdorff, the dual action of* $\hat{G}$ *on* $(A\times_\alpha G)^\wedge$ *is free and proper, and the restriction map* Res *of Proposition* 1.2 *induces a homeomorphism of* $(A\times_\alpha G)^\wedge/\hat{G}$ *onto* $\hat{A}$.

Let H be a compactly generated subgroup of G such that G/H is discrete, and recall that we can decompose $A\times_\alpha G$ as a restricted crossed product $(A\times_\alpha H)\times_{\beta|H} G$, where, because G is abelian,

$$\beta_s(f)(h)=\alpha_s(f(h)) \qquad \text{for} \quad f\in C_c(G, A, \mathscr{I}).$$

The results of [24, 26] show that $A\times_\alpha H$ is a continuous-trace algebra, and we aim to apply Proposition 1.5 to the H-inner system $(A \times_\alpha H, G, \beta)$. To make this work we need two simple lemmas.

LEMMA 1.8. *Suppose $\alpha: G \to \operatorname{Aut} A$ is pointwise unitary and H is central in G. Then $\beta: G \to \operatorname{Aut}(A \times_\alpha H)$ is pointwise unitary.*

Proof. Let $\pi \times V \in (A \times_\alpha H)^\wedge$; note that π is irreducible by Proposition 1.2. Let $U: G \to U(\mathscr{H}_\pi)$ be a representation which implements α in the representation π. Since $\operatorname{Ad} U|_H = \operatorname{Ad} V$, there exists $\gamma \in \hat{H}$ with $V = \gamma U|_H$, and because H is central

$$U_s^* V_h = U_s^* \gamma(h) U_h = \gamma(h) U_{s^{-1}h} = \gamma(h) U_{hs^{-1}} = V_h U_s^*.$$

Thus

$$\begin{aligned} \pi \times V(\beta_s(f)) &= \int \pi(\alpha_s(f(h))) V_h \, dh \\ &= U_s \left(\int \pi(f(h)) V_h \, dh \right) U_s^* \qquad (\text{since } U_s^* V_h = V_h U_s^*) \\ &= U_s \pi \times V(f) U_s^*, \end{aligned}$$

and U implements β in the representation $\pi \times V$.

LEMMA 1.9. *Suppose G is a locally compact group acting freely on a locally compact Hausdorff space P, and H is a closed normal subgroup of G. If H acts properly on P and G/H acts properly on $H \backslash P$, then G acts properly on P.*

Remark. Since H acts properly on P, $H \backslash P$ is a locally compact Hausdorff space [21], and G/H acts freely on $H \backslash P$, so it makes sense to assert that G/H acts properly.

Proof. Let K be a compact subset of P: we have to show that $\{s \in G: sK \cap K \neq \varnothing\}$ is relatively compact in G. Let K_1 be the (compact) image of K in $H \backslash P$, and let L_1 be a compact set in G/H such that

$$\{sH: (sH \cdot K_1) \cap K_1 = \varnothing\} \subseteq L_1. \tag{1}$$

Let L_2 be a compact subset of G such that $L_2/H \supseteq L_1$: we may as well assume L_2 is symmetric. We now set $K_2 = L_2 \cdot K$, so K_2 is compact in P, and choose a compact set L_3 in H such that

$$\{t \in H: tK_2 \cap K_2 \neq \varnothing\} \subseteq L_3. \tag{2}$$

Now suppose $s \in G$ and $sK \cap K \neq \varnothing$; say $k \in K$ satisfies $sk \in K$. Then

$(sH)(Hk) = Hsk \in K_1$, so $(sH)K_1 \cap K_1 \neq \emptyset$ and $sH \in L_1$ by (1). Thus we can find $t \in L_2$ such that $tH = sH$. Then $t^{-1}s \in H$ and

$$t^{-1}sk = t^{-1}s.e.k \in t^{-1}sK_2, \qquad t^{-1}sk \in t^{-1}K \subseteq L_2 \cdot K = K_2,$$

so $t^{-1}sK_2 \cap K_2 \neq \emptyset$. Thus by (2) we have $t^{-1}s \in L_3$, $s \in L_3 \subseteq L_2L_3$. We have now shown

$$\{s \in G: sK \cap K \neq \emptyset\} \subseteq L_2 L_3,$$

which is a compact subset of G, and this is what we wanted.

Proof of Theorem 1.7. Let H be a compactly generated subgroup of G such that G/H is discrete—for example, let K be a compact neighbourhood of e and take $H = \bigcup_{n=1}^{\infty} K^n$. By [29, Corollary 2.2], $\alpha|_H$ is locally unitary, and $\mathrm{Res}: (A \times_\alpha H)^\wedge \to \hat{A}$ is a locally trivial $\hat{H}$-bundle such that $A \times_\alpha H \cong \mathrm{Res}^* A$ [24, Theorem 2.2; 26, Proposition 1.5]. In particular, $A \times_\alpha H$ has continuous trace. The canonical map $\mathscr{I}$ of H into $M(A \times_\alpha H)$ is a twisting map for β, and there is an isomorphism of $A \times_\alpha G$ onto $(A \times_\alpha H) \times_{\beta|H} G$ such that the induced homeomorphism on spectra carries $\pi \times U \in (A \times_\alpha G)^\wedge$ into $(\pi \times U|_H) \times U$. (To see this, one can either chase through Green's construction of the isomorphism on p. 198 of [12], or verify that $(A \times_\alpha H) \times_{\beta|H} G$ has the universal property which characterises $A \times_\alpha G$ [20, Section 5].) The action β of G is pointwise unitary by Lemma 1.8, hence also pointwise unitary relative to $\mathscr{I}$ by Lemma 1.1. We can therefore apply Proposition 1.5 to deduce that $(A \times_\alpha G)^\wedge = ((A \times_\alpha H) \times_{\beta|H} G)^\wedge$ is Hausdorff, and that

$$\mathrm{Res}: \pi \times U \to (\pi \times U|_H) \times U \to \pi \times U|_H$$

induces a homeomorphism of $(A \times_\alpha G)^\wedge / H^\perp$ onto $(A \times_\alpha H)^\wedge$; notice that Res also converts the dual action of $\gamma \in \hat{G}$ into the dual action of $\gamma|_H \in \hat{H}$. Since $(A \times_\alpha H)^\wedge$ is a locally trivial $\hat{H}$-bundle, the action of $\hat{H}$ on $(A \times_\alpha H)^\wedge$ is in particular proper and hence so is the action of $\hat{G}/H^\perp$ on $(A \times_\alpha G)^\wedge/H^\perp$. The subgroup $H^\perp$ is compact, being the dual of the discrete group G/H, and therefore always acts properly. Thus Lemma 1.9 implies that the action of $\hat{G}$ on $(A \times_\alpha G)^\wedge$ is proper; it is free by Proposition 1.2, so this proves the theorem.

THEOREM 1.10. *Let (A, G, α) be a separable dynamical system with G abelian, α pointwise unitary and A a continuous-trace algebra with spectrum X. Let* $\mathrm{Res}: (A \times_\alpha G)^\wedge \to X$ *denote the restriction map and let τ denote the action of $\hat{G}$ on $C_0((A \times_\alpha G)^\wedge)$ induced by the dual action on $(A \times_\alpha G)^\wedge$. We denote by i_A the canonical embedding of A in $M(A \times_\alpha G)$, and view $C_0((A \times_\alpha G)^\wedge)$ as a subalgebra of $ZM(A \times_\alpha G)$, courtesy of the Dauns–*

Hofmann theorem. Then the map $\Psi(f \otimes a) = f i_A(a)$ *induces an isomorphism of the pull-back*

$$\mathrm{Res}^* A = C_0((A \times_\alpha G)^\wedge) \otimes_{C(X)} A$$

onto $A \times_\alpha G$, *which carries the action* $\mathrm{Res}^*(\mathrm{id}) = \tau \otimes_{C(X)} \mathrm{id}$ *into the dual action of* $\hat{G}$.

Proof. We shall in fact construct the inverse Φ of Ψ—it is a bit messier to write down, but easier to work with. We view A as the algebra $\Gamma_0(E)$ of sections of the C^*-bundle over $X = \hat{A}$ with fibre $E_x = A/\ker \varepsilon_x$; we write $a(x)$ for the image of $a \in A$ in E_x, and ε_x for the homomorphism $a \to a(x)$. For each irreducible representation $\pi \times V$ of $A \times_\alpha G$, there is a unique $x \in X$ such that $\pi \sim \varepsilon_x$, and a unique strictly continuous homomorphism $U: G \to UM(E_x)$ such that $\pi \times V \sim \varepsilon_x \times U$—to see this, just observe that π induces an isomorphism π_1 of E_x onto $\mathcal{K}(\mathcal{H}_\pi)$, and we can take U to be the homomorphism $\pi_1^{-1} \circ V$. The pull-back algebra $\mathrm{Res}^* A$ is canonically isomorphic to $\Gamma_0(\mathrm{Res}^* E)$ [26, Proposition 1.3], and we shall try to define $\Phi: A \times_\alpha G \to \Gamma_0(\mathrm{Res}^* E)$ by

$$\Phi(z)(\pi \times V) = \varepsilon_x \times U(z), \qquad \text{where} \quad \varepsilon_x \times U \sim \pi \times V.$$

Note that there are at least two problems here: it is not clear that $\Phi(z)$ is a continuous section nor that it vanishes at infinity. However, if we can establish these two facts, then Φ will automatically be an isometry, and hence extend to an isomorphism of $A \times_\alpha G$ into Γ_0. Once we have shown it is onto, it extends to an isomorphism of multiplier algebras; it is then a left inverse for the map Ψ in the statement of the theorem, which is therefore also an isomorphism.

To begin with, then, we have to show that Φ takes values in $\Gamma_0(\mathrm{Res}^* E)$. As a Banach space, $\Gamma_b(\mathrm{Res}^* E)$ is spanned by sections of the form

$$f \cdot a: \varepsilon_x \times U \to f(\varepsilon_x \times U) a(x) \qquad \text{for} \quad f \in C_0((A \times_\alpha G)^\wedge), \quad a \in \Gamma_b(E).$$

Thus to see that $\Phi(z)$ is a continuous section, it is enough to show that, for any $z \in C_c(G, A)$ of the form $\phi \otimes a \in C_c(G) \otimes A$, we can approximate $\Phi(\phi \otimes a)$ locally uniformly by a linear combination of sections of the form $f \cdot a$. Since the sections of compact support are dense in A, we may as well suppose $a \in \Gamma_c(E)$. Then at least $\Phi(\phi \otimes a)$ has support contained in $\mathrm{Res}^{-1}(\mathrm{supp}\, a)$, so that the question of whether $\Phi(\phi \otimes a)$ vanishes at infinity is also local in X. Let H be the subgroup of G generated by $\mathrm{supp}\, \phi$; since H is compactly generated, $\alpha|_H$ is locally unitary [29, Corollary 2.2]. As our current problems are local in X, we may suppose that $\alpha|_H$ is implemented by a strictly continuous homomorphism $u: G \to UM(A)$.

Now let $\varepsilon > 0$. We shall show that, under the above assumption,

(a) $[\pi \times V \in (A \times_\alpha G)^\wedge : \|\Phi(\phi \otimes a)(\pi \times V)\| \geqslant \varepsilon]$ is relatively compact

(b) there exists $\sum f_i \otimes b_i \in C_b((A \times_\alpha G)^\wedge) \otimes A$ such that

$$\left\| \sum f_i \otimes b_i - \Phi(\phi \otimes a) \right\| < \varepsilon.$$

Together, these show $\Phi(\phi \otimes a) \in \Gamma_0(\mathrm{Res}^* E)$.

Choose a finite open cover $\{N_i\}$ of supp ϕ, $r_i \in N_i$, such that

$$\|au_r - au_{r_i}\| < \varepsilon/2\ \|\phi\|_\infty\ \mu(\operatorname{supp}\phi) \qquad \text{for} \quad r \in N_i,$$

and let $\{\rho_i\}$ be a partition of unity subordinate to $\{N_i\}$. Provided ϕ is not identically zero—in which case (a) and (b) are trivially true—H is an open subgroup of G. The Haar measure on G therefore restricts to Haar measure on H and we have

$$\pi \times V(\phi \otimes a) = (\pi \times V|_H)(\phi \otimes a).$$

For any $(x, \gamma) \in X \times \hat{H}$ we have

$$\left\| \varepsilon_x \times \gamma u(x)(\phi \otimes a) - \sum_i (\phi\rho_i|_H)^\wedge\ (\gamma^{-1})(au_{r_i})(x) \right\| < \varepsilon/2. \tag{1}$$

Since a has compact support and each $(\phi\rho_i|_H)^\wedge$ vanishes at infinity,

$$\left\{(x, \gamma) : \left\| \sum_i (\phi\rho_i|_H)^\wedge\ (\gamma^{-1})(au_{r_i})(x) \right\| \geqslant \varepsilon/2 \right\} \tag{2}$$

is relatively compact in $X \times \hat{H}$. But by [24, Theorem 2.2], the map $(x, \gamma) \to \varepsilon_x \times \gamma u(x)$ is a homeomorphism of $T \times \hat{H}$ onto $(A \times_\alpha H)^\wedge$, and it follows from (1) and (2) that

$$\{\pi \times U \in (A \times_\alpha H)^\wedge : \|\pi \times U(\phi \otimes a)\| \geqslant \varepsilon\} \tag{3}$$

is relatively compact in $(A \times_\alpha H)^\wedge$. Since H is open in G, G/H is discrete and $H^\perp = (G|H)^\wedge$ is compact. The map $\pi \times V \to \pi \times V|_H$ is just the restriction map

$$\mathrm{Res}: (A \times_\alpha G)^\wedge = ((A \times_\alpha H) \times_{\beta|H} G)^\wedge \to (A \times_\alpha H)^\wedge$$

(see the proof of the previous theorem), which by Proposition 1.5 induces a homeomorphism of $(A \times_\alpha G)^\wedge/H^\perp$ onto $(A \times_\alpha H)^\wedge$—in particular, Res is proper. Thus (3) implies (a). To establish (b), we observe that

$$\begin{aligned} (A \times_\alpha G)^\wedge \to X \times \hat{H} \quad &\to (A \times_\alpha H)^\wedge \to \hat{H} \to \mathbf{C} \\ \varepsilon_x \times U \to \varepsilon_x \times U|_H & \\ = \varepsilon_x \times \gamma u(x) \to (x, \gamma) \quad &\to \gamma \ \to (\phi\rho_i|_H)^\wedge\ (\gamma) \end{aligned}$$

is a composition of continuous functions, and hence defines a continuous function $f_i \in C_b((A \times_\alpha G)^\wedge)$. Since $u_{r_i} \in M(A)$, $b_i = au_{r_i} \in A$ and (1) also implies (b).

We have now established that Φ is an isometric isomorphism of $A \times_\alpha G$ into $\Gamma_0(\mathrm{Res}^* E) \cong \mathrm{Res}^* A$. It is easy to check that Φ converts the action of $C_0((A \times_\alpha G)^\wedge)$ on $A \times_\alpha G$ into the natural action on $\mathrm{Res}^* A$, and hence to prove Φ is onto, it is enough to show that for each $\varepsilon_x \times U \in (A \times_\alpha G)^\wedge$, the set

$$\{\Phi(z)(\varepsilon_x \times U) \colon z \in A \times_\alpha G\}$$

is dense in the fibre E_x over $\varepsilon_x \times U$. But routine calculations show that if $\phi \in C_c(G)$ is a positive function with $\int \phi = 1$ and support near the identity, then $\Phi(\phi \otimes a)(\varepsilon_x \times U)$ is near $a(x)$ in E_x. Thus Φ is surjective, and, as we observed earlier, this proves that Ψ is an isomorphism. Finally, for $\gamma \in \hat{G}$ we have

$$\begin{aligned} \Phi(\hat{\alpha}_\gamma(z))(\varepsilon_x \times U) &= \varepsilon_x \times U(\hat{\alpha}_\gamma(z)) \\ &= \varepsilon_x \times \gamma^{-1} U(z) \\ &= \Phi(z)(\gamma^{-1} \cdot (\varepsilon_x \times U)) \\ &= \mathrm{Res}^*(\mathrm{id})_\gamma(\Phi(z))(\varepsilon_x \times U), \end{aligned}$$

and this completes the proof of Theorem 1.10.

COROLLARY 1.11. *Suppose α and β are two pointwise unitary actions of an abelian group on a continuous-trace algebra A. Then the $\hat{G}$-spaces*

$$\mathrm{Res}_\alpha \colon (A \times_\alpha G)^\wedge \to \hat{A}, \qquad \mathrm{Res}_\beta \colon (A \times_\beta G)^\wedge \to \hat{A}$$

are isomorphic if and only if α is exterior equivalent to β.

Proof. If $u \colon G \to UM(A)$ is an α 1-cocycle such that $\beta = \mathrm{Ad}\, u \circ \alpha$, then $\Phi(z)(s) = z(s)u_j^*$ extends to an isomorphism of $A \times_\alpha G$ onto $A \times_\beta G$ which intertwines the dual actions; in particular, the spectra are $\hat{G}$-isomorphic. Conversely, suppose there is a $\hat{G}$-equivariant homeomorphism h of $(A \times_\alpha G)^\wedge$ onto $(A \times_\beta G)^\wedge$ such that $\mathrm{Res}_\beta \circ h = \mathrm{Res}_\alpha$. Then h induces isomorphisms

$$h^*: C_0((A \times_\alpha G)^\wedge) \to C_0((A \times_\alpha G)^\wedge)$$
$$h^* \otimes \mathrm{id}: \mathrm{Res}_\beta^* A = C_0((A \times_\alpha G)^\wedge) \otimes_{C(\hat{A})} A \to \mathrm{Res}_\alpha^* A$$

which respect the actions τ and $\mathrm{Res}^*(\mathrm{id}) = \tau \otimes \mathrm{id}$ of $\hat{G}$ by translation. Composing with the isomorphisms Ψ_α, Ψ_β of the theorem gives an isomorphism

$$\Theta = \Psi_\alpha \circ (h^* \otimes \mathrm{id}) \circ \Psi_\beta^{-1}: A \times_\beta G \to A \times_\alpha G$$

which carries $\hat{\beta}$ into $\hat{\alpha}$. Further, $\Psi_\alpha(1 \otimes a) = i_A(a)$, and hence $\Theta(i_A(a)) = i_A(a)$ for $a \in A$. It therefore follows from [23, Theorem 35] (or [25, Theorem 0.10]) that α and β are exterior equivalent.

Remark. This proof of Corollary 1.11 is completely different from the one given in [24, Proposition 2.5] for the locally unitary case.

We have now seen that a pointwise unitary action of G is determined up to exterior equivalence by the $\hat{G}$-space $(A \times_\alpha G)^\wedge$. We shall now show that, if A is stable, every free and proper $\hat{G}$-space E with $E/\hat{G} = \hat{A}$ arises this way, so that studying pointwise unitary actions of G on A up to exterior equivalence is equivalent to studying free and proper $\hat{G}$-spaces with orbit space $\hat{A}$ up to $\hat{G}$-isomorphism. The idea of the proof is easy: given such a space E, the crossed product $C_0(E) \times \hat{G}$ is stably isomorphic to $C_0(E/\hat{G}, \mathscr{K})$ by a theorem of Green. Thus if α is the dual action of $G = \hat{\hat{G}}$, then by duality

$$G(E/\hat{G}, \mathscr{K}) \times_\alpha G \cong (C_0(E) \times \hat{G}) \times \hat{\hat{G}} \cong C_0(E) \otimes \mathscr{K},$$

and α has the required properties for $A = C_0(E/\hat{G}, \mathscr{K})$. For an arbitrary stable A, we write $A = A \otimes \mathscr{K} = A \otimes_{C(\hat{A})} C_0(\hat{A}, \mathscr{K})$ and take our action to be $\mathrm{id} \otimes \alpha$. The details of this construction were actually worked out in [15, Section 6] for the nonabelian case; however, we have noticed that coactions are not widely popular, so we shall also give a self-contained proof along the lines of [24, Section 3]. We shall need the following simple topological lemma:

LEMMA 1.12. *Let $q: E \to X$, $p: F \to Y$ be the orbit maps for two free, proper actions of a locally compact group G, and suppose $\phi: E \to F$ is a continuous G-equivariant surjection which induces a homeomorphism ψ of X onto Y. Then ϕ is a homeomorphism.*

Proof. Suppose that $\eta_\alpha \to \eta$ in F: we want to show there is a subnet such that $\phi^{-1}(\eta_{\alpha_\beta}) \to \phi^{-1}(\eta)$ in E. The continuity of p and ψ^{-1} imply that

$$q(\phi^{-1}(\eta_\alpha)) = \psi^{-1}(p(\eta_\alpha)) \to \psi^{-1}(p(\eta)) = q(\phi^{-1}(\eta)).$$

Since q is open, we can, by passing to a subnet, suppose there is a net $\{\xi_\alpha\}$ in E such that $q(\xi_\alpha)=q(\phi^{-1}(\eta_\alpha))$ and $\xi_\alpha\to\phi^{-1}(\eta)$. Let $g_\alpha\in G$ satisfy $g_\alpha\cdot\xi_\alpha=\phi^{-1}(\eta_\alpha)$. Then, on the one hand, we have

$$\phi(\xi_\alpha)\to\phi(\phi^{-1}(\eta))=\eta,$$

and on the other, we have

$$g_\alpha\cdot\phi(\xi_\alpha)=\phi(g_\alpha\cdot\xi_\alpha)=\eta_\alpha\to\eta.$$

If N is a compact neighbourhood of η, then eventually g_α belongs to $\{g\in G\colon gN\cap N\neq\varnothing\}$, which is relatively compact in G. Thus by passing to another subnet, we may suppose $g_\alpha\to g$. Then $g_\alpha\circ\phi(\xi_\alpha)\to g\cdot\eta$, and the freeness of the action forces $g=e$. We therefore have

$$\phi^{-1}(\eta_\alpha)=g_\alpha\cdot\xi_\alpha\to e\cdot\phi^{-1}(g)=\phi^{-1}(\eta),$$

which is what we wanted.

PROPOSITION 1.13. *Suppose A is a separable stable continuous-trace C^*-algebra with spectrum X, and $q\colon E\to X$ is the orbit map for a free and proper action of a separable locally compact abelian group G. Then there is a pointwise unitary action α of $\hat{G}$ on A and a G-equivariant homeomorphism h of $(A\times_\alpha\hat{G})^\wedge$ onto E such that* $\mathrm{Res}=q\circ h$.

Proof. We start with the dual action β of $\hat{G}$ on $C_0(E)\times G$. As in [24, Lemma 3.3] the map $\xi\to\operatorname{Ind}\varepsilon_\xi$ induces a homeomorphism ψ of E/G onto $\hat{B}$. We can realise $\operatorname{Ind}\varepsilon_\xi$ concretely on $L^2(G)$ via

$$[\operatorname{Ind}\varepsilon_\xi(z)x](s)=\int_G z(t,s\cdot\xi)x(t^{-1}s)\,dt$$

$$\text{for}\quad z\in C_c(G\times E),\quad x\in L^2(G).$$

If we define $U\colon\hat{G}\to U(L^2(G))$ by $U_\gamma x(s)=\overline{\gamma(s)}x(s)$, then $(\operatorname{Ind}\varepsilon_\xi, U)$ is covariant, and β is therefore pointwise unitary. As in [24, top of p. 230], the map $\xi\to(\operatorname{Ind}\varepsilon_\xi)\times U$ is a continuous bijection ϕ of E onto $(B\times_\beta\hat{G})^\wedge$ such that

$$\begin{array}{ccc} E & \xrightarrow{\phi} & (B\times_\beta\hat{G})^\wedge \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}} \\ E/G & \xrightarrow{\psi} & \hat{B} \end{array}$$

commutes; by Lemma 1.12, ϕ is a homeomorphism. By Green's theorem [11, Theorem 14], $C_0(E)\times G$ is the separable C^*-algebra defined by a continuous field $\mathscr{H}$ of Hilbert spaces over $X=E/G$, and then $(C_0(E)\times G)\otimes\mathscr{H}$

is naturally $C(X)$-isomorphic to the C^*-algebra defined by $\mathscr{H} \otimes (X \times H)$. But $\mathscr{H} \otimes (X \times H)$ is locally trivial by [4], hence trivial [5] and $(C_0(E) \times G) \otimes \mathscr{K}$ is $C_0(X)$ isomorphic to $C_0(X, \mathscr{K})$. Let γ be the action on $C_0(X, \mathscr{K})$ corresponding to $\beta \otimes \mathrm{id}$ on $(C_0(E) \times G) \otimes \mathscr{K}$: it is then pointwise unitary with spectrum G-isomorphic to E.

To finish off, write

$$A \cong A \otimes \mathscr{K} \cong A \otimes_{C(X)} C_0(X, \mathscr{K}) \qquad (C_0(X)\text{-isomorphisms}),$$

and take α to be the action on A corresponding to $\mathrm{id} \otimes \gamma$. Then

$$A \times_\alpha \hat{G} \cong (A \otimes_{C(X)} C_0(X, \mathscr{K})) \times_{\mathrm{id} \otimes \gamma} \hat{G} \cong A \otimes_{C(X)} (C_0(X, \mathscr{K}) \times_\gamma \hat{G}),$$

which has spectrum G-isomorphic to $(C_0(X, \mathscr{K}) \times \hat{G})^\wedge$ and hence to E. This completes the proof of Proposition 1.13.

We have now finished the proof of our main theorem: part (1) is established in Proposition 1.2, part (2) in Theorem 1.7 and Corollary 1.11, part (3) in Theorem 1.10, and part (4) in Proposition 1.13.

2. Generalisations

(a) *Actions Which Are Pointwise Unitary on a Subgroup*

After one understands automorphism groups $\alpha: G \to \operatorname{Aut} A$ which act trivially on $\hat{A}$, an obvious next step is to look at ones for which the action on $\hat{A}$ has a constant stabiliser H. Such actions were analyzed in [25, Section 2] under the hypotheses that $\alpha|_H$ is locally unitary and $\hat{A} \to \hat{A}/G$ locally trivial, and this had some surprising consequences (see [25, Section 4]). With the results we have obtained, it is quite easy to relax both these hypotheses.

Corollary 2.1. *Let α be an action of a locally compact abelian group G on a continuous-trace C^*-algebra A, and suppose that the induced action of G on $\hat{A}$ has constant isotropy group H. If $\alpha|_H$ is pointwise unitary and G/H acts properly on $\hat{A}$, then we have a commutative diamond of locally compact Hausdorff spaces*

$$\begin{tikzcd} & (A \times_\alpha H)^\wedge \arrow[dl, "\mathrm{Ind}"'] \arrow[dr, "\mathrm{Res}"] & \\ (A \times_\alpha G)^\wedge \arrow[dr, "q"'] & & \hat{A} \arrow[dl, "p"] \\ & \hat{A}/G & \end{tikzcd}$$

in which Res *and* q *are the orbit maps for free, proper actions of* $\hat{H}$ *and* Ind *and* p *are the orbit maps for free, proper actions of* G/H.

Proof. The existence of the analogous commutative diamond of primitive ideal spaces is established in [25, Proposition 2.1], and the argument used at the beginning of the proof of [25, Theorem 2.2] shows that under our hypotheses all the algebras are type I, so we can replace primitive ideal spaces by spectra. We have already seen that $(A \times_\alpha H)^\wedge$ is a locally compact Hausdorff space on which $\hat{H}$ acts freely and properly with orbit space $\hat{A}$. The action $\beta: G \to \operatorname{Aut} A \times_\alpha H$ given by $\beta_s(f)(t) = \alpha_s(f(t))$ induces a free action of G/H on $(A \times_\alpha H)^\wedge$, which commutes with the dual action of $\hat{H}$ and is converted by the restriction map Res into the given action on $\hat{A}$. To see this action of G/H on $(A \times_\alpha H)^\wedge$ is proper, it is enough to show that if $s_i \cdot x_i \to y$ and $x_i \to x$ in $(A \times_\alpha H)^\wedge$, then $\{s_i\} \subseteq G/H$ has a convergent subnet. But Res is continuous, so

$$(s_i(\operatorname{Res} x_i), x_i) = (\operatorname{Res}(s_i \cdot x_i), \operatorname{Res} x_i) \to (\operatorname{Res} y, \operatorname{Res} x),$$

and the properness of $\hat{A}$ as a G/H-space gives the required subnet. To see that the dual action of $\hat{G}/H^\perp = \hat{H}$ on $(A \times_\alpha G)^\wedge$ is proper, we can use the argument in the second paragraph of the proof of [27, Theorem 6.3] essentially verbatim. Finally, that Ind and p induce homeomorphisms of the orbit spaces $(A \times_\alpha H)^\wedge/G$, $(A \times_\alpha G)^\wedge/\hat{H}$ onto $(A \times_\alpha G)^\wedge$, $\hat{A}/G$ are proved in [25, Proposition 2.1] and [9, Corollary 2.5], respectively.

Concluding Remark. It is tempting to wonder whether there is a version of this result for actions with continuously varying stabilisers, along the lines of [27, Theorem 6.3] but without the "locally unitary" hypothesis; in particular, is the spectrum of the stabiliser algebra $A \times_\alpha \mathscr{P}$ of [27] always Hausdorff? Our present methods rely heavily on the result of Rosenberg which guarantees that pointwise unitary actions of compactly generated groups are locally unitary. So far, however, we only have an analogue of this in the case of varying stabilisers when locally they are all isomorphic to a fixed compactly generated subgroup [27, Proposition 5.5]. Although the examples in [27] show that this does happen quite often, this means that any extension of Corollary 3.1 obtained using our present methods will require some fairly clumsy-looking hypotheses. It could be interesting, therefore, to find a non-structure-theoretic proof of our main result.

(b) *Pointwise Unitary Twisted Actions*

Let (α, u) be a twisted action of G on A in the sense of [20]. We shall say (α, u) is *pointwise unitary* if for each $\pi \in \hat{A}$ there is a Borel map $U: G \to U(H_\pi)$ such that (π, U) is a covariant representation of (A, G, α, u)

(see [20 Definition 2.3]), in which case we say U implements (α, u) in the representation π. Similarly, (α, u) is *locally unitary* if for each $\pi \in \hat{A}$, there is a strictly Borel map $w: G \to UM(A)$ such that $(\rho, \rho \circ w)$ is a covariant representation for ρ in some neighbourhood of π in $\hat{A}$. Our present aim is to extend our earlier results on pointwise unitary actions to twisted actions using the stabilisation trick of [20, Section 3]. We shall need the following:

LEMMA 2.2. *Suppose G is abelian. Then there is an action $\hat{\alpha}$ of $\hat{G}$ on $A \times_{\alpha, u} G$ (called the* dual action*) such that*

$$\hat{\alpha}_\gamma(i_A(a) i_G(z)) = i_A(a) i_G(\bar{\gamma} z) \qquad \text{for} \quad a \in A, \quad z \in L^1(G).$$

Proof. Write $B = A \times_{\alpha, u} G$, and define $j_\gamma: G \to UM(B)$ by $j_\gamma(s) = \gamma(s) i_G(s)$. We claim that the triple (B, i_A, j_γ) is also a crossed product for (A, G, α, u). First of all, it is easy to check that (i_A, j_γ) is covariant. Next, observe that $(\pi, \gamma U)$ is covariant if (π, U) is, and

$$(\pi \times \gamma U) \circ j_\gamma(s) = \overline{\gamma(s)}(\gamma U)(s) = U_s,$$

so condition (b) of [20, Definiton 2.4] holds. To check condition (c) just note that

$$i_A \times j_\gamma(z) = \int_G i_A(z(s)) \overline{\gamma(s)} i_G(s)\, ds = i_A \times i_G(\bar{\gamma} z),$$

and that $z \to \bar{\gamma} z$ is an isometry of $L^1(G, A)$ onto itself. Therefore by the uniqueness of the crossed product [20, Proposition 2.7] there is an automorphism $\hat{\alpha}_\gamma$ of B which does the required things to generators. Finally, if $z \in L^1(G, A)$ we have

$$\hat{\alpha}_\gamma(i_A \times i_G(z)) = i_A \times i_G(\bar{\gamma} z),$$

and the continuity of $\hat{\alpha}$ follows from the continuity of the map $\gamma \to \bar{\gamma} z: \hat{G} \to L^1(G, A)$.

Suppose (α, u) is a twisted action of G on A. According to [20, Theorem 3.4], the twisted action $(\alpha \otimes \mathrm{id}, u \otimes 1)$ of G on $A \otimes \mathscr{K}$ is exterior equivalent to a genuine action β of $A \otimes \mathscr{K}$; say $v: G \to UM(A \otimes \mathscr{K})$ is the (α, u)-cocycle such that $\beta = \mathrm{Ad}\, v \circ (\alpha \otimes \mathrm{id})$. It follows from [20, Lemma 3.3] that if (π, U) is a covariant representation of (A, G, α, u) then $(\pi \otimes \mathrm{id}, \pi \otimes \mathrm{id}(v)(U \otimes 1))$ is a covariant representation of $(A \otimes \mathscr{K}, G, \beta)$; since

$$(A \otimes \mathscr{K})^\wedge = \{\pi \otimes \mathrm{id}: \pi \in \hat{A}\},$$

this shows that β is pointwise unitary if (α, u) is. Similarly, if $w: G \to UM(A)$ implements (α, u) near $\pi \in \hat{A}$, then $s \mapsto v_s(w_s \otimes 1)$

implements β near $\pi\otimes\mathrm{id}\in(A\otimes\mathscr{K})^\wedge$, so β is locally unitary if (α, u) is. By [20, Lemma 3.3], the exterior equivalence induces an isomorphism

$$\phi:(A\times_{\alpha,u}G)\otimes\mathscr{K}\cong(A\otimes\mathscr{K})\times_{\alpha\otimes\mathrm{id},u\otimes 1}G\to(A\otimes\mathscr{K})\times_\beta G,$$

satisfying

$$\begin{aligned}\phi(i_A(a)\otimes 1)&=i_{A\otimes\mathscr{K}}(a\otimes 1)\\ \phi(i_G(s)\otimes 1)&=i_{A\otimes\mathscr{K}}(v_s^*)i_G(s).\end{aligned}\qquad(*)$$

This in turn induces a homeomorphism $\hat{\phi}$ of $(A\times_{\alpha,u}G)^\wedge=((A\times_{\alpha,u}G)\otimes\mathscr{K})^\wedge$ onto $((A\otimes\mathscr{K})\times_\beta G)^\wedge$ such that

$$\hat{\phi}(\pi\times U)=(\pi\otimes\mathrm{id})\times(\pi\otimes\mathrm{id}(v)(U\otimes 1)).$$

Now suppose G is abelian, (α, u) is pointwise unitary (so that β is too), and A has continuous trace. Then a representation $\pi\times U$ of $A\times_{\alpha,u}G$ is irreducible iff $(\pi\otimes\mathrm{id})\times(\pi\otimes\mathrm{id}(v)U\otimes 1)$ is irreducible, hence by Proposition 1.2 iff $\pi\otimes\mathrm{id}$ is irreducible, hence iff π is irreducible. Thus we have a well-defined restriction map Res: $(A\times_{\alpha,u}G)^\wedge\to\hat{A}$ such that

$$\begin{tikzcd}(A\times_{\alpha,u}G)^\wedge \arrow[rr, "\hat{\phi}"] \arrow[dr, "\mathrm{Res}"'] & & ((A\otimes\mathscr{K})\times_\beta G)^\wedge \arrow[dl, "\mathrm{Res}"] \\ & \hat{A} & \end{tikzcd}$$

commutes. It follows easily from $(*)$ that ϕ intertwines the dual actions of $\hat{G}$, and $\hat{\phi}$ therefore becomes an isomorphism of $\hat{G}$-spaces. We can therefore read off information about the $\hat{G}$-space $(A\times_{\alpha,u}G)^\wedge$ from Theorem 1.10.

COROLLARY 2.3. *Suppose (A, G, α, u) is a separable twisted dynamical system with G abelian, A continuous trace and (α, u) pointwise unitary. Then a representation $\pi\times U$ of $A\times_{\alpha,u}G$ is irreducible iff π is irreducible, and the resulting restriction map* Res: $(A\times_{\alpha,u}G)^\wedge\to\hat{A}$ *is a continuous open surjection. The spectrum $(A\times_{\alpha,u}G)^\wedge$ is a locally compact Hausdorff space, the dual action of $\hat{G}$ on $(A\times_{\alpha,u}G)^\wedge$ is free and proper, and* Res *induces a homeomorphism of the orbit space $(A\times_{\alpha,u}G)^\wedge/\hat{G}$ onto $\hat{A}$. Further, $A\times_{\alpha,u}G$ is a continuous-trace algebra with $\delta(A\times_{\alpha,u}G)=\mathrm{Res}^*\,\delta(A)$.*

Remarks. (1) Presumably we still have $A\times_{\alpha,u}G\cong\mathrm{Res}^*A$, but because our present methods involve stabilisation this would require more work. We shall not need this here, since for our application to the group algebra $C^*(G)$ we are primarily interested in the topology on the spectrum $\hat{G}$.

(2) Every restricted crossed product $A \times_{\alpha|N} G$ of Green is isomorphic to a twisted crossed product $A \times_{\alpha,u} G/N$ [20, Section 5], and it is trivial to verify that if the action of G is pointwise unitary relative to the twist $\mathscr{I}$, then the corresponding twisted action (α, u) is pointwise unitary. Hence we can apply this result directly to Green's algebras.

(c) *Pointwise Unitary Actions on Algebras with Hausdorff Spectrum*

The results of [24, 26] show that our main theorem holds also for locally unitary actions on C^*-algebras with Hausdorff spectrum, whereas we have insisted throughout that our C^*-algebras have continuous trace. Now we have certainly used this hypothesis repeatedly—in particular, every time we used Rosenberg's result [29, Corollary 2.2], which is not valid even for actions of $\mathbf{Z}$ if the algebra does not have continuous trace (see the example below). It is natural to wonder, however, if this hypothesis is really necessary or just a limitation of our present methods. The following example shows that it is indeed necessary.

EXAMPLE 2.4. Let $A = \{a \in C([0,1], M_2(\mathbf{C})) \colon a(0) \in \mathbf{C}1\}$, $W = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We define an automorphism ϕ of A by $\phi(a)(x) = Wa(x)W^*$, and take $\alpha \colon \mathbf{Z} \to \operatorname{Aut} A$ to be the group generated by $\phi = \alpha_1$. Then α is pointwise unitary and A has Hausdorff spectrum $[0, 1]$, but $(A \times_\alpha \mathbf{Z})^\wedge$ is not Hausdorff.

Proof. Let ε_r denote the irreducible representation $a \to a(r)$ of A. Then α is implemented in the representation ε_r, for $r > 0$, by the unitary representation $U \colon n \to W^n$, and in ε_0 by the trivial representation of $\mathbf{Z}$. Thus, if γ_z denotes the character $n \to z^n$ of Z, we can describe $(A \times_\alpha \mathbf{Z})^\wedge$ setwise as

$$\{\varepsilon_r \times \gamma_z U : r \in (0, 1], z \in S^1\} \cup \{\varepsilon_0 \times \gamma_z \colon z \in S^1\}.$$

We claim that $\varepsilon_{1/n} \times U$ converges to both $\varepsilon_0 \times \gamma_1$ and $\varepsilon_0 \times \gamma_{-1}$ in $(A \times_\alpha \mathbf{Z})^\wedge$. To see this, it will be enough to show that every subbasic neighbourhood of $\varepsilon_0 \times \gamma_{\pm 1}$ eventually contains $\varepsilon_{1/n} \times U$; since $C_c(\mathbf{Z}, A)$ is dense in $A \times \mathbf{Z}$ and every function in $C_c(\mathbf{Z}, A)$ is a sum of elements of the form $\delta_m a$ for $m \in \mathbf{Z}$, $a \in A$, we need only consider neighbourhoods of the form

$$\begin{aligned} N_j = \{\pi \times V \in (A \times_\alpha Z)^\wedge : \exists \eta \in \mathscr{H}_{\pi \times V}, \|\eta\| \leqslant 1 \text{ such that} \\ |(\pi \times V(\delta_m a)\eta \mid \eta) - \varepsilon_0 \times \gamma_j(\delta_m a)| < \varepsilon\}, \end{aligned}$$

for $j = \pm 1$. Choose n large enough to ensure $\|a(1/n) - a(0)\| < \varepsilon$, so that, in particular, the diagonal entries $a(1/n)_{11}$, $a(1/n)_{22}$ both differ from the complex number $a(0)$ by less than ε. Then with $\eta = (1, 0)$ we have

$$\begin{aligned}(\varepsilon_{1/n}\times U(\delta_m a)\eta\,|\,\eta) &= (\varepsilon_{1/n}(a)\,U_m(\eta)\,|\,\eta)\\ &= (a(1/n)\,W^m\eta\,|\,\eta)\\ &= (a(1/n)\eta\,|\,\eta)\\ &= a(1/n)_{11},\\ \varepsilon_0\times\gamma_1(\delta_m a) &= \varepsilon_0(a)\gamma_1(m)=a(0)^m=a(0),\end{aligned}$$

so $\varepsilon_{1/n}\times U\in N_1$. Similarly, with $\eta=(0,1)$ we have

$$\begin{aligned}(\varepsilon_{1/n}\times U(\delta_m a)\eta\,|\,\eta) &= (a(1/n)\,W^m\eta\,|\,\eta)\\ &= (-1)^m\,(a(1/n)\eta\,|\,\eta)=(-1)^m\,a(1/n)_{22},\end{aligned}$$

which differs from $\varepsilon_0\times\gamma_{-1}(\delta_m a)=a(0)(-1)^m$ by less than ε, and $\varepsilon_{1/n}\times U$ also belongs to N_{-1}.

Remark. A more detailed analysis shows that the spectrum of $A\times_\alpha\mathbf{Z}$ can be identified with $[0,1]\times S^1$, topologised as follows: $(0,1]\times S^1$, $\{0\}\times S^1$ have the usual topology, and the union is topologised so that sequences in $(0,1]\times S^1$ converging to $(0,z)$ in the usual topology converge to both $(0,z)$ and $(0,-z)$ in $(A\times_\alpha\mathbf{Z})^\wedge$.

3. Applications

(a) *A Characterisation of Crossed Products with Continuous Trace*

Suppose a locally compact group G acts freely on a locally compact space X. An elegant theorem of Green [11, Theorem 17] asserts that G acts properly if and only if the transformation group C^*- algebra $C_0(X)\times G$ has continuous trace. We shall now use our main theorem to prove a similar result for crossed products $A\times G$, where A is a continuous trace algebra with spectrum X and G is abelian. Our proof uses duality and appears to be completely different from Green's.

THEOREM 3.1. *Let $\alpha\colon G\to\operatorname{Aut}A$ be an action of a separable locally compact abelian group on a separable continuous-trace C^*-algebra, and suppose G acts freely on the spectrum of A. Then $A\times_\alpha G$ has continuous trace if and only if G acts properly on $\hat{A}$.*

The following lemma must be well known, although we lack a specific reference; further, it is almost certainly true in much greater generality. We thank Colin Sutherland for a suggestion which shortened our original proof; the argument is modelled on Williams' proof of the analogous fact for transformation group algebras [31, Proposition 4.2].

LEMMA 3.2. *Suppose* (A, G, α) *are as in Theorem* 3.1. *Then for each* $\pi \in \hat{A}$, $\operatorname{Ind}_{\{e\}}^G \pi$ *is an irreducible representation of* $A \times_\alpha G$.

Proof. Let $X = \hat{A}$, and view A as the algebra $\Gamma_0(E)$ of sections of a C^*-bundle E over X. We write $\varepsilon_x \colon A \to E_x \cong \mathcal{K}(\mathcal{H})$ for evaluation at $x \in X$. Then $\operatorname{Ind} \varepsilon_x$ acts in $L^2(G, \mathcal{H})$ via $\tilde{\varepsilon}_x \times \lambda$, where λ is the left regular representation of G and $\tilde{\varepsilon}_s(a)\xi(s) = \alpha_s^{-1}(a)(\xi(s))$. To show $\operatorname{Ind} \varepsilon_x$ irreducible, it will be enough to show that

$$\operatorname{Ind} \varepsilon_x(A \times_\alpha G)'' \supseteq M(C_0(G, \mathcal{K})) \cup \lambda(G) \otimes 1,$$

where M is the representation of $C_0(G, \mathcal{K})$ on $L^2(G, \mathcal{H})$ by pointwise multiplication. Since $(\operatorname{Ind} \varepsilon_x)'' = (\tilde{\varepsilon}_x \times \lambda)''$ obviously contains $\lambda(G) \otimes 1$, it will be enough to show $M(C_0(G, \mathcal{K})) \subseteq \tilde{\varepsilon}_x(A)''$.

We next observe that if $\Phi \colon A \to C_b(G, E_x) \cong C_b(G, \mathcal{K})$ is defined by $\Phi(a)(s) = \alpha_s^{-1}(a)(x)$, then $\tilde{\varepsilon}_x = M \circ \Phi$. Thus it will be enough for us to show that we can approximate $\phi \in C_0(G, \mathcal{K})$ strictly in $M(C_0(G, \mathcal{K}))$ by elements in the range of Φ; for if $\Phi(a_n) \to \phi$ strictly, then

$$\tilde{\varepsilon}_x(a_n) = M \circ \Phi(a_n) \to M(\phi) \qquad \text{strongly}$$

and $M(\phi) \in \tilde{\varepsilon}_x(a_n)''$. In order to construct the a_n's we shall need some notation. We define isomorphisms $\alpha_{s,x} \colon E_{s^{-1}x} \to E_x$ by $\alpha_{s,x}(a(s^{-1}x)) = \alpha_s(a)(x)$; routine arguments show that $\alpha_{s,x}$ is well defined and that $\alpha_{s,x}$ is an isomorphism with inverse $\alpha_{s^{-1}, s^{-1}x}$. We now claim that if K is compact in G then

$$s \cdot x \to (\alpha_{s^{-1},x})^{-1}\,(\phi(s))$$

is a continuous section of $E|_{K \cdot x}$.

We fix $s_0 \in K$ and choose $a \in \Gamma_0(E)$ such that $a(x) = \phi(s_0)$. Then

$$(\alpha_{s^{-1},x})^{-1}\,(\phi(s_0)) = \alpha_{s,sx}(a(x)) = \alpha_s(a)(sx).$$

The map $s \to s \cdot x$ is a homeomorphism of K onto $K \cdot x$ (it is crucial that K is compact here), and so $s \cdot x \to s \to \alpha_s(a)$ is continuous from $K \cdot x$ to A. Hence

$$(s \cdot x, t \cdot x) \to \alpha_s(a)(t \cdot x)$$

is a continuous section of $(K \cdot x) \times E|_{K \cdot x}$, and the restriction of this to the diagonal is a continuous section of $E|_{K \cdot x}$. Now if $a, b \in A$ then

$$\|\alpha_s(a)(s \cdot x) - \alpha_s(b)(s \cdot x)\| = \|\alpha_{s,sx}((a-b)(x))\| = \|(a-b)(x)\|,$$

so we have

$$\|\alpha_s(\phi(s))(s\cdot x)-\alpha_s(\phi(s_0))(s\cdot x)\|<\varepsilon \qquad \text{whenever} \quad \|\phi(s)-\phi(s_0)\|<\varepsilon.$$

In other words, our section $s\cdot x\to\alpha_s(\phi(s))(s\cdot x)$ can be locally uniformly approximated near $s_0\cdot x$ by the continuous section $s\cdot x\to\alpha_s(\phi(s_0))(s\cdot x)$, and hence is itself a continuous section.

We now choose an increasing sequence of compact subsets K_n of G with $\bigcup K_n=G$. We can extend each section

$$s\cdot x\in K_n\cdot x\to(\alpha_{s^{-1}},\ x)(\phi(s))$$

to an element a_n of A, and then for $s\in K_n$

$$\Phi(a_n)(s)=\alpha_s^{-1}(a_n)(x)=\alpha_{s^{-1},x}(a_n(s\cdot x))=\phi(s).$$

Thus, in particular, $\Phi(a_n)\to\phi$ uniformly on compact in G, and hence in $M(C_0(G,\mathscr{K}))$. This completes the proof of Lemma 3.2.

Proof of Theorem 3.1. The "if" direction is [25, Theorem 1.1], so suppose $A\times_\alpha G$ has continuous-trace. The lemma implies that $\operatorname{Ind}\pi$ is an irreducible representation of $A\times_\alpha G$ for each $\pi\in\hat{A}$, and in fact all the irreducible representations of $A\times_\alpha G$ have this form. For by the Gootman–Rosenberg theorem [10], every primitive ideal $I=\ker\rho$ of $A\times_\alpha G$ is induced from a primitive ideal $\ker\pi$ of A; then $I=\operatorname{Ind}(\ker\pi)=\ker(\operatorname{Ind}\pi)$ and, since $A\times_\alpha G$ is type I, we must have $\rho\sim\operatorname{Ind}\pi$. We can realise $\operatorname{Ind}\pi=\tilde{\pi}\times\lambda$ in $L^2(G,\mathscr{H}_\pi)$, as in the proof of the lemma:

$$[\operatorname{Ind}\pi(z)\xi](s)=\int\pi(\alpha_s^{-1}(z(t)))(\xi(t^{-1}s))\,dt \qquad \text{for} \quad z\in C_c(G,A).$$

If we define $M\colon\hat{G}\to U(L^2(G,\mathscr{H}_\pi))$ by $M_\gamma\xi(s)=\overline{\gamma(s)}\xi(s)$, then a quick calculation shows

$$M_\gamma\operatorname{Ind}\pi(z)M_\gamma^*\xi=\operatorname{Ind}\pi(\hat{\alpha}_\gamma(z))\xi \qquad \text{for} \quad z\in C_c(G,A),\quad \xi\in L^2(G,\mathscr{H}_\pi);$$

in other words, $(\operatorname{Ind}\pi, M)$ is covariant, and $\hat{\alpha}$ is pointwise unitary.

It follows from Theorem 1.6 that $((A\times_\alpha G)\times_{\hat{\alpha}}\hat{G})^\wedge$ is Hausdorff and that, via the double dual action $\hat{\hat{\alpha}}$, $G=\hat{\hat{G}}$ acts freely and properly on $((A\times_\alpha G)\times_{\hat{\alpha}}\hat{G})^\wedge$. By Takesaki–Takai duality, there is a covariant isomorphism of

$$((A\times_\alpha G)\times_{\hat{\alpha}}\hat{G},\hat{\hat{\alpha}}) \qquad \text{onto} \qquad (A\otimes\mathscr{K}(L^2(G)),\alpha\otimes\operatorname{Ad}\rho)$$

(see, for example, [22, 7.9.3]), so we can deduce that, via $\alpha \otimes \mathrm{Ad}\,\rho$, G acts freely and properly on $(A \otimes \mathscr{K})^\wedge$. But the homeomorphism $\pi \to \pi \otimes i$ of $\hat{A}$ with $(A \otimes \mathscr{K})^\wedge$ intertwines the original action of G with the one inherited from $\alpha \otimes \mathrm{Ad}\,\rho$, so the original action must have been proper too.

(b) *The Dual Topology for a Locally Compact Group*

Suppose G is a second countable locally compact group and N a closed normal subgroup which is type I and regularly embedded in G. Then, according to the Mackey machine [16], every irreducible representation can be obtained by choosing $L \in \hat{N}$, extending it to a multiplier representation $\hat{L}$ of the stabiliser

$$H_L = \{s \in G : s \cdot L \sim L, \text{ where } s \cdot L(n) = L(s^{-1}ns)\}$$

in such a way that the multiplier σ_L is constant on N-cosets in H_L, choosing an irreducible $\bar{\sigma}_L$-representation M of H_L which is also constant on N-cosets, and forming $\mathrm{Ind}_{H_L}^G \hat{L} \otimes M$. The equivalence class then depends only on the orbit $G \cdot L$ of L in $\hat{N}$ and the class of M in $(H_L/N, \bar{\sigma}_L)^\wedge$. Thus if $\mathscr{L}$ is any G-invariant subset of $\hat{N}$, we can consider the subset $\hat{G}(\mathscr{L})$ of $\hat{G}$ consisting of the representations associated to orbits lying in $\mathscr{L}$.

If now $\mathscr{L}$ is a closed G-invariant subset of $\hat{N}$, the spectrum of the ideal $I = \bigcap \{\ker L : L \in \mathscr{L}\}$ can be canonically identified with $\hat{N} \backslash \mathscr{L}$, and that of $C^*(N)/I$ with $\mathscr{L}$. If we decompose $C^*(G)$ as $C^*(N) \times_{\alpha|N} G$ [12, Proposition 1], then $I \times_{\alpha|N} G$ embeds naturally as an ideal of $C^*(G)$ with quotient isomorphic to $(C^*(N)/I) \times_{\alpha|N} G$ [12, Proposition 12] and the spectrum of $(C^*(N)/I) \times_{\alpha|N} G$ is then naturally identified with $\hat{G}(\mathscr{L})$. In fact, if $\mathscr{L}$ is just locally closed we can perform a similar construction: take $I = \bigcap \{\ker L : L \in \bar{\mathscr{L}}\}$, $J = \bigcap \{\ker L : L \in \bar{\mathscr{L}} \backslash \mathscr{L}\}$, so that the subquotient J/I has spectrum $(J/I)^\wedge = (\hat{N} \backslash (\bar{\mathscr{L}} \backslash \mathscr{L})) \backslash (\hat{N} \backslash \mathscr{L}) = \mathscr{L}$, and then $((J/I) \times_{\alpha|N} G)^\wedge$ is naturally homeomorphic to $\hat{G}(\mathscr{L})$. This observation is fundamental to Green and Rieffel's version of the Mackey machine [28, 12, 13].)

Schochetman's program [30] for describing the topology on $\hat{G}$ involves splitting $\hat{N}$ into saturated subsets $\mathscr{L}$ according to the behaviour of the stabilisers, determining the topology on $\hat{G}(\mathscr{L})$ and then trying to fit the pieces together. Our results give no information about this last part of his program, but they should shed some light on his proposed description of bits of $\hat{G}(\mathscr{C})$, where $\mathscr{C} = \{L \in \hat{N} : H_L = G\}$. To use our methods to get positive results towards his conjectures, we shall have to suppose that G/N is abelian and that the subquotients $A(\mathscr{L})$ corresponding to $\mathscr{L} \subseteq \hat{N}$ are continuous-trace algebras. While Schochetman himself frequently makes the first assumption, the second is certainly more restrictive, although there are plenty of groups N for which $C^*(N)$ has continuous trace, and since N

is type I, $C^*(N)$ will always have substantial continuous-trace subquotients. In any case, our chief aim here is to suggest some modifications to his conjectures.

Schochetman begins his analysis of $\hat{G}(\mathscr{C})$ by considering the subset $\mathscr{C}_1$ of $\mathscr{C}$ consisting of those $L \in \mathscr{C}$ for which the Mackey obstruction σ_L is trivial. He first of all conjectures that $\mathscr{C}_1$ is closed, and we shall later show that this is the case under our hypotheses provided G/N is also compactly generated. Suppose this is true for now. For $L \in \mathscr{C}_1$, the representation L of N extends to an ordinary representation of G—in other words, to a representation which implements the action of G on $C^*(N)$ relative to the twist $\mathscr{I}$ in the representation L of $C^*(N)$. Thus the action of G on the quotient $A(\mathscr{C}_1)$ is pointwise unitary relative to the twist $\mathscr{I}$, and by our Theorem, $\hat{G}(\mathscr{C}_1)^\wedge = (A(\mathscr{C}_1) \times_{\alpha|N} G)^\wedge$ is a locally compact Hausdorff space, the dual action of $(G/N)^\wedge$ on $\hat{G}(\mathscr{C}_1)^\wedge$ is free and proper, and the orbit space is homeomorphic to $\mathscr{C}_1$. Conjecture D_2 of [30] asserts that this space should be homeomorphic to $(\mathscr{C}_1 \times (G/N))^\wedge$, but of course from our point of view this is rather unlikely. Indeed, if we take $G = \mathbf{R}$ and $N = \mathbf{Z}$, the action of G on $\hat{N} = \mathbf{T}$ is trivial and $\mathscr{C}$ is therefore all of $\hat{N}$. The cohomology group $H^2(G/N, \mathbf{T}) = H^2(\mathbf{T}, \mathbf{T})$ is trivial, so in fact $\mathscr{C}_1 = \hat{N}$. But $\hat{G}$ is homeomorphic to $\mathbf{R}$, whereas $\mathscr{C}_1 \times (G/N)^\wedge = \mathbf{T} \times \mathbf{Z}$. Of course, this is a silly way to try to compute the topology on $\hat{\mathbf{R}}$, but our results suggested looking at it because $\mathbf{R} \to \mathbf{T}$ is a classical example of a non-trivial principal bundle. In any case, it suggests that conjecture D_2 should be amended to say that $\hat{G}(\mathscr{C}_1)$ is a free and proper $(G/N)^\wedge$-space with orbit space $\mathscr{C}_1$, locally trivial if G/N is compactly generated.

Remark. Unfortunately this counterexample to conjecture D_2 of [30] also contradicts [30, Proposition 3.13]. The problem occurs when it is asserted that the extension map $L \in \mathscr{C}_1 \subseteq \hat{N} \to \hat{L} \in \hat{G}$ is continuous; in our example this would amount to giving a continuous map of $\mathbf{T} = \hat{N}$ into $\mathbf{R} = \hat{G}$ which was a section for the covering map of $\mathbf{R}$ onto $\mathbf{R}/\mathbf{Z} = \mathbf{T}$.

The next step in the program of [30] is to analyse the remainder $\mathscr{C}_2 = \mathscr{C} \backslash \mathscr{C}_1$ of $\mathscr{C}$. The only case considered there is the one where $(G/N, \bar{\sigma}_L)^\wedge$ is a singleton for each $L \in \mathscr{C}_2$—equivalently, where each symmetriser

$$S_L = \{s \in G/N : \sigma_L(s, t) = \sigma_L(t, s) \text{ for all } t \in G/N\}$$

is trivial [2]. However, we suggest that one could also partition $\mathscr{C}_2$ into the sets $\mathscr{C}_w$ where the Mackey obstruction $w \in H^2(G/N, \mathbf{T})$ is constant, and try to describe $\hat{G}(\mathscr{C}_w)$. The following lemma will be useful in this regard and may be of some independent interest. In it, $\mathbf{H}^2(G, \mathbf{T})$ denotes the Moore cohomology group endowed with the quotient topology inherited from the natural Polish topology on $Z^2(G, \mathbf{T}) \subseteq C^2(G, \mathbf{T})$.

LEMMA 3.3. *Suppose $(A, G, \alpha, \mathscr{I})$ is a separable twisted covariant system with A continuous trace, G/N compactly generated and α acting trivially on $X = \hat{A}$. Then the map which sends $x \in X$ to the Mackey obstruction at x is continuous from X to $\mathbf{H}^2(G/N, \mathbf{T})$.*

Proof. We follow the proof of [29, Corollary 2.2]. Since this is a local problem, we may as well suppose X is compact. Then $\operatorname{Inn} A$ is open in $\operatorname{Aut}_{C(X)} A$ [25, Theorem 0.8], so the quotient $\operatorname{Out}_{C(X)} A$ is discrete. Since G acts trivially on X, we have $\alpha(G) \subseteq \operatorname{Aut}_{C(X)} A$ and $\alpha(N) \subseteq \operatorname{Inn} A$. The image of $\alpha(G)$ in $\operatorname{Out}_{C(X)} A$ is therefore a compactly (and hence finitely) generated subgroup. Let $\{\alpha_i\}$ be a finite set in $\operatorname{Aut}_{C(X)} A$ whose images in $\operatorname{Out}_{C(X)} A$ generate this subgroup. Because $C(X)$-automorphisms are locally inner, by shrinking X one may suppose that each α_i is inner, and then α consists entirely of inner automorphisms. The obstruction to implementing α by a unitary group $u: G \to UM(A)$ extending τ is then given by the class in $H^2(G/N, C(X, \mathbf{T}))$ of the cocycle $w \in Z^2(G/N, C(X, \mathbf{T}))$ described in Proposition 1.5. Evaluating this cocycle at a point $x \in X$ gives a cocycle $w(x) \in Z^2(G/N, \mathbf{T})$ which represents the Mackey obstruction class $c(x)$. Now each $w(s, t)$ is continuous; so if $x_n \to x$ then $w(x_n) \to w(x)$ pointwise, and hence by [17, Proposition 6] in the Polish group $\mathbf{Z}^2(G/N, \mathbf{T})$. The topology on $\mathbf{H}^2$ is the quotient of this topology, so the result follows.

COROLLARY 3.4. *If w is closed in $\mathbf{H}^2(G/N, \mathbf{T})$ then*

$$\mathscr{C}_w = \{x \in X : c(x) = w\}$$

is closed in X.

When G/N is abelian, $\mathbf{H}^2(G/N, \mathbf{T})$ is Hausdorff [18, Theorem 7], and this corollary applies to each $w \in H^2(G/N, \mathbf{T})$. Therefore provided G/N is compactly generated all the sets $\mathscr{C}_w$ are closed in $\mathscr{C}$, and hence the corresponding sets $\hat{G}(\mathscr{C}_w)$ are closed in $\hat{G}(\mathscr{C})$. Thus in particular conjecture D_1 of [30] (the case $w = 0$) is valid under our hypotheses; this observation seems to be almost completely distinct from Schochetman's own result [30, Proposition 3.12], which applies only when $\mathscr{C}_1$ consists of one-dimensional representations.

To handle the sets $\hat{G}(\mathscr{C}_w)$, we observe that, as in the preceding section, the twisted action (α, u) of G/N on the quotient $A = A(\mathscr{C}_w)$ is stably exterior equivalent (via a cocycle v, say) to a genuine action β of G/N on $A \otimes \mathscr{K}$. We claim that w is the Mackey obstruction for the action β at each point of $\mathscr{C}_w = \hat{A}$. To see this, suppose $\pi \in \mathscr{C}_w$ and $U: G \to U(\mathscr{H})$ is a w-representation such that $U|_N = \pi \circ \mathscr{I}$ and U implements α in the representation π. If (π, U) were an ordinary representation of $(A, G, \mathscr{I})$, the corresponding covariant representation of $(A, G/N, \alpha, u)$ would be $(\pi, U \circ c)$ [20, proof of Proposition 5.1], and the corresponding representation of

$(A \otimes \mathscr{K}, G/N, \beta)$ would be $(\pi \otimes \mathrm{id}, \pi \otimes \mathrm{id}(v)((U \circ c) \otimes 1))$ [20, Lemma 3.3]. Thus we try

$$W_{sN} = \pi \otimes \mathrm{id}(v_{sN})(U(c(s)) \otimes 1).$$

We trivially have

$$W_{sN} \pi \otimes \mathrm{id}(b) W_{sN}^* = \pi \otimes \mathrm{id}(\beta_{sN}(b)) \qquad \text{for} \quad sN \in G/N,$$

and we can compute:

$$\begin{aligned}
W_{sN} W_{tN} &= \pi \otimes \mathrm{id}(v_{sN})(U(c(s)) \otimes 1) \pi \otimes \mathrm{id}(v_{tN})(U(c(t)) \otimes 1) \\
&= \pi \otimes \mathrm{id}(v_{sN}) \pi \otimes \mathrm{id}(\alpha_{c(s)}(v_{tN}))((U(c(s))\, U(c(t))) \otimes 1) \\
&= \pi \otimes \mathrm{id}(v_{stN}(u(sN, tN)^* \otimes 1))(w(sN, tN)\, U(c(s)c(t)) \otimes 1) \\
&= w(sN, tN) \pi \otimes \mathrm{id}(v_{stN}(u(sN, tN)^* \otimes 1)) \\
&\qquad \times ((U(c(s)c(t)c(st)^{-1})\, U(c(st))) \otimes 1) \\
&= w(sN, tN) \pi \otimes \mathrm{id}(v_{stN})(U(c(st)) \otimes 1)) \\
&\qquad\qquad \text{since} \quad U|_N = \pi \circ \mathscr{I} \quad \text{and} \quad u(sN, tN) = \mathscr{I}(c(s)c(t)c(st)^{-1}) \\
&= w(sN, tN)\, W_{stN}.
\end{aligned}$$

Thus W is a w-representation of G/N which implements β in the repesentation π—i.e., the Mackey obstruction at π is w, as claimed.

Now let

$$S = \{sN \in G/N : w(sN, tN) = w(tN, sN) \text{ for all } tN \in G/N\}$$

be the symmetriser of w. Then by [14, Theorem 1.1], restriction gives a homeomorphism of $((A \otimes \mathscr{K}) \times_\beta G/N)^\wedge$ onto $((A \otimes \mathscr{K}) \times_\beta S)^\wedge$, which is easily seen to be $(G/N)^\wedge$-equivariant. Putting this together with the homeomorphism of $(A \times_{\alpha|N} G)^\wedge$ onto $((A \otimes \mathscr{K}) \times_\beta G/N)^\wedge$ induced by the exterior equivalence shows that $\mathrm{Res}: (A \times_{\alpha|N} G)^\wedge \to \hat{A}$ is $(G/N)^\wedge$-isomorphic to $((A \otimes \mathscr{K}) \times_\beta S)^\wedge$. The action of S on $A \otimes \mathscr{K}$ is pointwise unitary, so we can apply our main results to it: we deduce that $(A \times_{\alpha|N} G)^\wedge$ is a locally compact Hausdorff space; that the dual action of $(G/N)^\wedge$ has common isotropy group $S^\perp$; and that the resulting action of $\hat{S} = \hat{G}/S^\perp$ is proper with $(A \times_{\alpha|N} G)^\wedge/\hat{S} = \hat{A} = \mathscr{C}_w$. Thus our amended version of conjecture D_2 of [30] seems equally reasonable for each of the sets $\hat{G}(\mathscr{C}_w)$; predicting that $\hat{G}(\mathscr{C}_w)$ is a free and proper $\hat{S}$-space over $\mathscr{C}_w$.

Concluding Remark. We have only discussed the conjectures of Schochetman concerning the part of $\hat{G}$ where induction is not a necessary step in the Mackey machine. However, we believe the phenomena we have described will arise elsewhere in this program. The space $\mathscr{F}(G, \mathscr{L})$ of irreducible representations of the stabilisers H_L for $L \in \mathscr{L}$ is topologized by viewing it as a subset of the spectrum of Fell's subgroup algebra $C^*(\mathscr{S})$ [6]. If $\mathscr{L}$ is G-invariant, there is a natural action of G on $\mathscr{F}(G, \mathscr{L})$ and Schochetman suggests that if $H_L \neq G$ for all $L \in \mathscr{L}$, then $\hat{G}(\mathscr{L})$ should be homeomorphic to $\mathscr{F}(G, \mathscr{L})/G$. The space $\mathscr{F}(G, \mathscr{L})$ is naturally fibred over $\mathscr{L}$ with fibre over L homeomorphic to $(H_L/N, \bar{\sigma}_L)^\wedge$, and our results indicate that this space could have non-trivial topological properties. For example, if $H_L = H \neq G$ for all $L \in \mathscr{L}$, then $\mathscr{F}(G, \mathscr{L})$ can be identified with $(A(\mathscr{L}) \times_{\alpha|N} H)^\wedge$ and this could easily be a nontrivial $(H/N)^\wedge$-bundle over $\mathscr{L}$. (See [27] for an idea of what this might mean if the stabilisers H_L vary continuously.) It does seem at present, though, that our understanding of the topology on the spectrum of a crossed product (e.g. [25, 27]) is not sufficient to give any new information about the induction step: if the stabilisers vary continuously, the conjectures on $\hat{G}$ were established years ago by Fell (see [1, Section 3]).

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